Determinants related to Dirichlet characters modulo 2, 4 and 8 of binomial coefficients and the algebra of recurrence matrices

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Abstract: Using recurrence matrices, defined and described with some details, we study a few determinants related to evaluations of binomial coefficients on Dirichlet characters modulo $2, 4$ and 8. [1]

1 Introduction

The aim of this paper is twofold: It contains computations of a few determinants related to binomial coefficients. The most interesting example, discovered after browsing through [7], is obtained by considering binomial coefficients modulo 4. We include two similar examples taken from [2] and [3].

The second topic discussed in this work are recurrence matrices, see [3] for a very condensed outline. They are defined as certain sequences of matrices involving self-similar structures and they form an algebra. Computations in the algebra of recurrence matrices are the main tool for proving the determinant formulae mentionned above. Recurrence matrices are however of independent interest since they are closely linked for example to automatic sequences, see [1], to rational formal power series in free noncommuting variables and to groups of automata, see [9] for the perhaps most important example. The determinant calculations of this paper can thus be considered as illustrations of some interesting features displayed by recurrence matrices.

The sequel of this paper is organised as follows:

The next section recalls mostly well-known facts concerning binomial and $q-$binomial coefficients and states the main results.

Section 3 defines the algebra $\mathcal{R}$ of recurrence matrices. It describes them with more details than necessary for proving the formulae of Section 2.

Section 4 discusses a few features of the group of invertible elements in $\mathcal{R}$.

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Section 5 proves formulae for the determinant of the reduction modulo 2 of the symmetric Pascal matrix (already contained in [2]) and of a determinant related to the 2−valuation of the binomial coefficients (essentially contained in [3]).

Section 6 is devoted to the proof of our main result, a formula for $\det(Z(n))$ where $Z(n)$ is the matrix with coefficients $\chi_B(\binom{s+t}{s})\in\{0,\pm1\}$, $0\leq s,t<n$ obtained by considering the "Beeblebrox reduction" (given by the Dirichlet character $\chi_B(2m)=0$, $\chi_B(4m\pm1)=\pm1$ for $m\in\mathbb{Z}$) of binomial coefficients. The proof uses an LU factorisation of the infinite symmetric matrix $Z=Z(\infty)$ and suggests to consider two (perhaps interesting) groups Γ_L and Γ_Z whose generators display beautiful "self-similar" structures. This section ends with a short digression on the "lower triangular Beeblebrox matrix" and the associated group.

Section 7 contains some data concerning the reduction of binomial coefficients by a Dirichlet character modulo 8 related to the Jacobi symbol.

Section 8 reproves a known formula for evaluating $q−$binomial coefficients at roots of unity. This formula yields easily formulae for some determinants associated to the reduction modulo 2 and the Beeblebrox reduction of (real and imaginary parts) of $q−$binomial coefficients evaluated at $q=-1$ and $q=i$.

2 Main results

2.1 Reductions modulo 2

Let $P(n)$ be the integral symmetric $n\times n$ matrix with coefficients $P_{s,t}\in\{0,1\}$, $0\leq s,t<n$ defined by

$$P_{s,t}\equiv\binom{s+t}{s}\pmod 2$$

where $\binom{s+t}{s}=\frac{(s+t)!}{s!\,t!}$ denotes the usual binomial coefficient involved in the expansion $(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$.

The evaluation $\binom{s+t}{s}\pmod 2$ can be computed using a Theorem of Lucas, see [11], page 52. Given a prime number p, it states that

$$\binom{n}{k}\equiv\prod_{j\geq0}\binom{\nu_j}{\kappa_j}\pmod p$$

where $\nu_i,\kappa_i\in\{0,1,\ldots,p-1\}$ are the coefficients of the $p−$ary expansion of $n=\sum_{j\geq0}\nu_jp^j$ and $k=\sum_{j\geq0}\kappa_jp^j$. Another formula (due to Kummer) for $\binom{n}{k}\pmod 2$ will be presented in Section 2.2.

Let $ds(n)=\sum_{j=0}^{\lfloor log_2(n)\rfloor}\nu_j\in\mathbb{N}$ denote the digit-sum of a natural integer with binary expansion $n=\sum_{j\geq0}\nu_j2^j$, $\nu_0,\nu_1,\cdots\in\{0,1\}$.

Theorem 2.1. *We have*

$$\det(P(2n)) = (-1)^n$$

and

$$\det(P(2n+1)) = (-1)^{n+ds(n)} .$$

Remark 2.2. *The infinite symmetric integral matrix $\tilde{P}$ with coefficients $\tilde{P}_{s,t} = \binom{s+t}{s}$ given by the binomial coefficients is sometimes called the Fermat matrix. Vandermonde's identity $\sum_{k=0}^{s} \binom{s}{k}\binom{t}{k} = \sum_{k=0}^{s} \binom{s}{k}\binom{t}{t-k} = \binom{s+t}{t}$ shows that $\det(\tilde{P}(n)) = 1$ where $\tilde{P}(n)$ is the symmetric $n \times n$ submatrix with coefficients $\binom{s+t}{s}, 0 \le s, t < n$ of $\tilde{P}$.*

2.2   2−valuations

Given a prime p, we denote by $v_p : \mathbb{Q}^* \longrightarrow \mathbb{N}$ the p−valuation. Any rational number α can thus be written in the form $\alpha = p^{v_p(\alpha)} \frac{n}{m}$ with $n, m \in \mathbb{Z}$ coprime to p. Let $V(n)$ be the symmetric $n \times n$ matrix with coefficients $V_{s,t} \in \{\pm 1, \pm i\}$ given by

$$V_{s,t} = i^{v_2(\binom{s+t}{s})}, \ 0 \le s, t < n .$$

The p−valuation $v_p(\binom{s+t}{s})$ of a binomial coefficient can be computed using a Theorem of Kummer stating that $v_p(\binom{s+t}{s})$ equals the number of carries occuring during the addition of the p−ary integers $s = \sum_{j \ge 0} \sigma_j p^j$ and $t = \sum_{j \ge 0} \tau_j p^j$. More precisely, Kummer shows the identity

$$v_p(x!) = \frac{1}{p-1}\left(x - \sum_{j \ge 0} \xi_j\right)$$

(see Lehrsatz, page 115 of [10]) where $x = \sum_{j \ge 0} \xi_j p^j \in \mathbb{N}$ with $\xi_j \in \{0, 1, \dots, p-1\}$. This implies the formula

$$v_p\left(\binom{s+t}{s}\right) = v_p((s+t)!) - v_p(s!) - v_p(t!) = \frac{1}{p-1}\sum_{j \ge 0}(\sigma_j + \tau_j - u_j)$$

(see [10], page 116) where $\sigma_j, \tau_j, u_j \in \{0, 1, \dots, p-1\}$ are defined by the p−ary expansions $s = \sum_{j \ge 0} \sigma_j p^j, t = \sum_{j \ge 0} \tau_j p^j$ and $s + t = \sum_{j \ge 0} u_j p^j$.

The next result uses the regular folding sequence $f : \{1, 2, \dots\} \longrightarrow \{\pm 1\}$. It is defined recursively by $f(2^n) = 1$ and $f(2^n + a) = -f(2^n - a)$ for $1 \le a < 2^n$, see for example [1].

Theorem 2.3. *We have*

$$\det(V(2n)) = (-1)^n \prod_{k=1}^{2n-1}(1 - f(k)i) \in \mathbb{Z}[i]$$

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and

$$\det(V(2n+1)) = (-1)^{n+ds(n)} \prod_{k=1}^{2n}(1 - f(k)i) \in \mathbb{Z}[i]$$

(with $ds(\sum_{j=0} \nu_j 2^j) = \sum_{j \geq 0} \nu_j$ denoting the binary digit-sum).

Remark 2.4. *Let D denote the diagonal matrix with diagonal entries $i^{ds(0)}, i^{ds(1)}, i^{ds(2)}, \ldots$. The paper [3] deals with the Hankel matrix H defined by $H_{s,t} = i^{ds(s+t)}$, $0 \leq s,t$ related by $H = D\overline{V}D$ to the complex conjugate $\overline{V}$ of the matrix V involved in Theorem 2.3.*

Let us also mention that slight extensions of the computations occuring in our proof of Theorem 2.3 establish the existence of nice continued $J-$fraction expansions for the formal power series (cf. [3])

$$\prod_{k=0}^{\infty}(1 + ix^{2^k}) \ ,$$

$$\frac{1}{x}\left(\frac{1+i}{2} + \frac{1-x}{i-1}\prod_{k=0}^{\infty}(1 + ix^{2^k})\right) \ ,$$

$$\frac{1}{x^2}\left(\frac{1+i}{2} + \frac{i-1}{2}x + \frac{1-x^2}{i-1}\prod_{k=0}^{\infty}(1 + ix^{2^k})\right) \ .$$

2.3   Beeblebrox reduction

The idea (and the "Beeblebrox" terminology) of considering the "Beeblebrox reduction" of binomial coefficients are due to Granville, see [7] and [8].

We define the "*Beeblebrox reduction*" as the Dirichlet character $\chi_B :$ $\mathbb{Z} \longrightarrow \{0, \pm 1\}$ given by

$$\chi_B(x) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod 2 \\ 1 & \text{if } x \equiv 1 \pmod 4 \\ -1 & \text{if } x \equiv 3 \pmod 4 \end{cases}$$

or equivalently by $\chi_B(2\mathbb{Z}) = 0, \chi_B(4\mathbb{Z} \pm 1) = \pm 1$. Beeblebrox reduction is the unique Dirichlet character modulo 4 not factorising through $\mathbb{Z}/2\mathbb{Z}$.

The following result allows fast computations of $\chi_B(\binom{n}{k})$.

Theorem 2.5. *We have*

$$\begin{aligned}
\chi_B(\tbinom{2n}{2k}) &= \chi_B(\tbinom{n}{k}) \\
\chi_B(\tbinom{2n}{2k+1}) &= 0 \\
\chi_B(\tbinom{2n+1}{2k}) &= (-1)^k \, \chi_B(\tbinom{n}{k}) \\
\chi_B(\tbinom{2n+1}{2k+1}) &= (-1)^{n(k+1)} \, \chi_B(\tbinom{n}{k})
\end{aligned}$$

We denote by $Z(n)$ (where the letter Z stands for Zaphod Beeblebrox, following the amusing terminology of [7] and [8]) the symmetric *Beeblebrox*

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matrix of size $n \times n$ with coefficients $Z_{s,t} \in \{-1, 0, 1\}$ for $0 \le s, t < n$ given by the Beeblebrox reduction $Z_{s,t} = \chi_B(\binom{s+t}{s})$ of binomial coefficients.

Define $f : \mathbb{N} \longrightarrow \pm 3^{\mathbb{Z}}$ by $f(0) = 1, f(1) = -1$ and recursively by

$$f(2^a + b) = \begin{cases} 3f(b) & \text{if } 2b < 2^a \\ \frac{1}{3}f(b) & \text{otherwise} \end{cases}$$

for $n = 2^a + b \ge 2$ where $0 \le b < 2^a$. Laurent Bartholdi pointed out that the value $f(n)$ of a binary integer $n = \sum_{i \ge 0} \nu_i 2^i$ is also given by

$$\begin{aligned} f(n) &= (-1)^n 3^{\sharp\{i \ | \ \nu_i = 0, \ \nu_{i+1} = 1\} - \sharp\{i \ | \ \nu_i = \nu_{i+1} = 1\}} \\ &= (-1)^n \prod_{i \ge 0} 3^{(1 - 2\nu_i)\nu_{i+1}} . \end{aligned}$$

Theorem 2.6. *We have*

$$\det(Z(n)) = \prod_{k=0}^{n-1} f(k) \in \pm 3^{\mathbb{N}} .$$

2.4 The Jacobi-character modulo 8

Let $\chi_J : \mathbb{N} \longrightarrow \{0, \pm 1\}$ denote the Dirichlet character modulo 8 defined by $\chi_J(2\mathbb{Z}) = 0$ and

$$\chi_J(n) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod 8 , \\ -1 & \text{if } n \equiv \pm 3 \pmod 8 . \end{cases}$$

Since $\chi_J(p) \equiv 2^{(p-1)/2} \pmod 2$ for p an odd prime, we call χ_J the Jacobi-character and we consider the matrix $J(n)$ with coefficients $J_{s,t} = \chi_J(\binom{s+t}{s})$ for $0 \le s, t < n$. The techniques of this paper can be used to prove the following result:

Theorem 2.7. *We have*

$$\det(J(n)) = \prod_{k=0}^{n-1} g(k) \in \pm 3^{\mathbb{N}}$$

where

$$g(n) = (-1)^n \prod_{k \ge 0} 3^{e(\lfloor n/2^k \rfloor)}$$

with $e(k)$ the $8-$periodic function given by the table

k	0	1	2	3	4	5	6	7
$e(k)$	0	0	1	-1	0	-2	3	-1

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The matrices $Z(n)$ and $J(n)$ share many features. However the considerable greater complexity of all objects attached to J makes it more difficult to pin down interesting algebraic structures associated to J.

Let me also add that the remaining Dirichlet character modulo 8 (given by $\tilde{\chi}(2\mathbb{Z}) = 0$ and $\tilde{\chi}(\epsilon a + 8\mathbb{Z}) = \epsilon$ for $a \in \{1, 3\}$ and $\epsilon \in \{\pm 1\}$) does not seem to give something interesting: the symmetric infinite matrix obtained by applying $\tilde{\chi}$ to binomial coefficients has probably no LU decomposition in the algebra of recurrence matrices.

Similarly, there seem to be no new interesting Dirichlet characters (mod 16) (with values in the Gaussian integers $\{\pm 1, \pm\sqrt{-1}\}$).

Remark 2.8. *The case of the character* $n \longmapsto \chi(n) = \epsilon_n \equiv n^{(p-1)/2}$ *(mod p) with* $\epsilon_n \in \{0, \pm 1\}$ *for p an odd prime gives rise to similar results which are somewhat trivial. Indeed, Lucas's theorem implies that the corresponding infinite matrix with coefficients* $\chi(\binom{i+j}{i})$, $0 \leq i, j$ *has a structure of an infinite tensor-power (corresponding to a "recurrence matrix of complexity 1"). It is thus not very interesting and easy to handle.*

The same remark holds for the remaining characters (mod p) when working in a suitable field (or integral subring) of cyclotomic numbers.

2.5 $q-$binomials

The expansion $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}$ involving two non-commuting variables x, y related by $yx = qxy$ where q is a central variable defines the $q-$binomials coefficients

$$\binom{n}{k}_q = \frac{\prod_{j=1}^{n}(1 - q^j)}{\left(\prod_{j=1}^{k}(1 - q^j)\right)\left(\prod_{j=1}^{n-k}(1 - q^j)\right)} \in \mathbb{N}[q] \ .$$

An ordinary binary coefficient $\binom{s+t}{s}$ can be identified with the number of lattice paths with steps $(1, 0)$ and $(0, -1)$, starting at $(0, s)$ and ending at $(t, 0)$. Similarly, the coefficient of q^c in the $q-$binomial $\binom{s+t}{s}_q$ counts the number of such paths delimiting a polygon of area c in the first quadrant $\{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\}$.

Reflecting all paths contributing to $\binom{s+t}{s}_q$ with respect to the diagonal line $x = y$ yields the equality

$$\binom{s+t}{s}_q = \binom{s+t}{t}_q \ .$$

Rotating all paths contributing to $\binom{s+t}{s}_q$ by a half-turn centered at $\frac{1}{2}(t, s)$ shows the identity

$$\binom{s+t}{s}_q = q^{st}\binom{s+t}{s}_{q^{-1}} \ .$$

Partitioning all paths contributing to $\binom{s+t}{s}_q$ accordingly to the nature of their first step (horizontal or vertical) shows the recursive formula

$$\binom{s+t}{s}_q = q^s \binom{s+t-1}{s}_q + \binom{s+t-1}{s-1}_q$$

or equivalently $\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$ which is the $q-$version of the celebrated recurrence relation $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for ordinary binomial coefficients.

Cutting all lattice paths γ contributing to $\binom{s+t}{s}_q$ along the diagonal line $s = t$ in two lattice paths shows the formula

$$\sum_k q^{k^2} \binom{s}{k}_q \binom{t}{k}_q = \binom{s+t}{s}_q$$

where $k \in \{0, 1, \ldots, min(s,t)\}$. This identity amounts to the matrix identity $P_q = L_q D_q L_q^t$ where P_q is the infinite symmetric matrix with coefficients $\binom{s+t}{s}_q$, $0 \le s, t$, where L_q is the lower triangular unipotent matrix with coefficients $\binom{s}{t}_q$, $0 \le s, t$ and where D_q is diagonal with diagonal coefficients $1, q, q^4, q^9, q^{16}, q^{25}, \ldots$. Denoting by $P_q(n)$ the submatrix $\binom{s+t}{s}_q$, $0 \le s, t < n$ formed by the first n rows and columns of P_q we have the identity $\det(P_q(n)) = q^{\sum_{j=0}^{n-1} j^2}$ which specialises to the identity $\det(P_1(n)) = 1$ of Remark 2.2. Appendix I of [6] contains many more formulae for $\binom{n}{k}_q$.

The formula of Lucas

$$\binom{a}{b} \equiv \binom{\lfloor a/p \rfloor}{\lfloor b/p \rfloor} \binom{a \pmod{p}}{b \pmod{p}} \pmod{p}$$

(where p is a prime number and where $a \pmod p, b \pmod p \in \{0, 1, \ldots, p-1\}$), see Section 2.1 or [11], has the following known analogue for $q-$binomials which reduces their evaluation at roots of 1 of small order to evaluations of ordinary binomial coefficients.

Theorem 2.9. *If $\omega = e^{2i\pi k/n}$ is a primitive $n-$th root of 1 (ie. $(k, n) = 1$ with $k \in \mathbb{Z}$ and $n \in \mathbb{N}$) then*

$$\binom{a}{b}_\omega = \binom{\lfloor a/n \rfloor}{\lfloor b/n \rfloor}_1 \binom{a \pmod{n}}{b \pmod{n}}_\omega$$

for all $a, b \in \mathbb{N}$ where $a \pmod n, b \pmod n \in \{0, 1, \ldots, n-1\}$.

Theorem 2.9 can be used to establish formulae for determinants of the symmetric matrices obtained by considering the reduction modulo 2, the Beeblebrox reduction or the reduction using the Jacobi character modulo 8 of (the real and imaginary part of) $\binom{s+t}{s}_q$, $0 \le s, t < n$ evaluated at $q = -1$ and $q = i$.

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3 The algebra of recurrence matrices

Recurrence matrices, introduced in [3], are a convenient tool for proving our main results. Recurrence matrices are closely related to rational formal power series in free non-commutative variables and can be considered as generalisations of finite state automata or of iterated tensor products. They arise also naturally in the context of "automata groups", a notion generalising a famous group of Grigorchuk, see [9]. The following exposition does not strive for exhaustivity or for the largest possible generality. Generalisations (e.g. by replacing the field of complex numbers by an arbitrary commutative field or by considering sequences of square matrices of size $k^n \times k^n, n \in \mathbb{N}$ for $k \in \{1, 2, 3, \dots\}$) are fairly straightforward and contained in [3] or with more details in [4].

The papers [5] and [12] deal with interesting subalgebras, called self-similar algebras, formed by recurrence matrices.

3.1 Recurrence matrices

Consider the vector space

$$\mathcal{A} = \prod_{n=0}^{\infty} M_{2^n \times 2^n}(\mathbb{C})$$

whose elements are sequences $A = (A[0], A[1], A[2], \dots)$ with $A[n] \in M_{2^n \times 2^n}(\mathbb{C})$ denoting a complex square matrix of size $2^n \times 2^n$. The obvious product

$$AB = (A[0]B[0], A[1]B[1], A[2]B[2], \dots)$$

turns $\mathcal{A}$ into an associative algebra. Denoting by

$$\rho(0,0)A, \rho(0,1)A, \rho(1,0)A, \rho(1,1)A \in \mathcal{A}$$

the four "corners" of

$$A = A[0], \left(\begin{array}{cc} (\rho(0,0)A)[0] & (\rho(0,1)A)[0] \\ (\rho(1,0)A)[0] & (\rho(1,1)A)[0] \end{array} \right), \left(\begin{array}{cc} (\rho(0,0)A)[1] & (\rho(0,1)A)[1] \\ (\rho(1,0)A)[1] & (\rho(1,1)A)[1] \end{array} \right), \dots$$

obtained (after deletion of the 1×1 matrix $A[0]$) by considering for all $n \geq 1$ the $2^{n-1} \times 2^{n-1}$ submatrix defined by the first or last 2^{n-1} rows and by the first or last 2^{n-1} columns of $A[n]$, we get four linear endomorphisms $\rho(s,t) \in \operatorname{End}(\mathcal{A})$, $0 \leq s, t \leq 1$, of the vector space $\mathcal{A}$. We call these endomorphisms *shift maps*. Using a hopefully suggestive synthetic notation, an element $A \in \mathcal{A}$ can thus be written as

$$A = A[0], \left(\begin{array}{cc} \rho(0,0)A & \rho(0,1)A \\ \rho(1,0)A & \rho(1,1)A \end{array} \right)$$

with $A[0] \in \mathbb{C}$ and $\rho(0,0)A, \rho(0,1)A, \rho(1,0)A, \rho(1,1)A \in \mathcal{A}$.

Definition A subspace $\mathcal{V} \subset \mathcal{A}$ is *recursively closed* if $\rho(s,t)\mathcal{V} \subset \mathcal{V}$ for all s, t.

The *recursive closure* $\overline{\mathcal{S}}$ of a subset $\mathcal{S} \in \mathcal{A}$ is the smallest recursively closed subspace of $\mathcal{A}$ which contains $\mathcal{S}$. We denote by $\overline{A}$ the recursive closure of the subset $\{A\}$ reduced to a single element $A \in \mathcal{A}$. The *complexity* of $A \in \mathcal{A}$ is the dimension $dim(\overline{A}) \in \mathbb{N} \cup \{\infty\}$ of the recursive closure $\overline{A} \subset \mathcal{A}$.

An element $A \in \mathcal{A}$ is a *recurrence matrix* if its recursive closure $\overline{A}$ is of finite dimension. We denote by $\mathcal{R} \subset \mathcal{A}$ the subset of all recurrence matrices.

Writing $\rho(X_{s,t})A$ or simply $X_{s,t}A$ for $\rho(s,t)A$, $0 \leq s, t \leq 1$, the shift maps $\rho(s,t) \in \mathrm{End}(\mathcal{A})$ induce a linear representation (still denoted) $\rho : \{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^* \longrightarrow \mathrm{End}(\mathcal{A})$, recursively defined by

$$(X_{s_1,t_1} X_{s_2,t_2} \cdots X_{s_l,t_l})A = (X_{s_1,t_1} X_{s_2,t_2} \cdots X_{s_{l-1},t_{l-1}})(\rho(s_l,t_l)A) \ ,$$

of the free non-commutative monoid $\{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^*$, called the *shift monoid*, in four generators $X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}$ representing shift maps. Subrepresentations of ρ correspond to recursively closed subspaces $\mathcal{V}$ of $\mathcal{A}$ spanned by (unions of) orbits under $\{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^*$.

The linear action of the monoid $\{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^*$ on $\mathcal{A}$ suggests to consider the bijective map which associates an element $A \in \mathcal{A}$ with the non-commutative formal power series

$$\sum_{\mathbf{X} \in \{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^*} ((\mathbf{X}A)[0])\mathbf{X} \in \mathbb{C}\langle\!\langle X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1} \rangle\!\rangle$$

in four free non-commutative variables $X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}$. This bijection restricts to a bijection between the vector space $\mathcal{R}$ of recurrence matrices and rational elements in $\mathbb{C}\langle\!\langle X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1} \rangle\!\rangle$.

The algebraic structure of $\mathcal{R} \subset \mathcal{A}$ is described by the following result.

Proposition 3.1. *(i) We have $dim(\overline{\lambda A}) = dim(\overline{A})$ for all $\lambda \in \mathbb{C}^*$ and for all $A \in \mathcal{A}$.*

(ii) We have $dim(\overline{A+B}) \leq dim(\overline{A}) + dim(\overline{B})$ for all $A, B \in \mathcal{A}$.

(iii) We have $dim(\overline{AB}) \leq dim(\overline{A}) \, dim(\overline{B})$ for all $A, B \in \mathcal{A}$.

Remark 3.2. *The inequalities of assertion (ii) and (iii) can of course be strict: Consider two elements $A, B \in \mathcal{A}$ defined by $A[n] = \frac{1+(-1)^n}{2}\mathrm{Id}[n]$, $B[n] = \frac{1-(-1)^n}{2}\mathrm{Id}[n]$ where $\mathrm{Id}[n]$ denotes the identity matrix of size $2^n \times 2^n$. The elements A, B have common recursive closure $\overline{A} = \overline{B} = \mathbb{C}A + \mathbb{C}B$ of dimension 2. Their sum $A + B = \mathrm{Id} \in \mathcal{R}$ is the identity element having complexity 1 and their product $AB = 0$ has complexity 0.*

Corollary 3.3. *The set $\mathcal{R}$ of recurrence matrices is a subalgebra of $\mathcal{A}$.*

Proof of Proposition 3.1 (i) and (ii) are obvious.

Denoting (slightly abusively) by $\overline{A}\,\overline{B} = \{\sum X_i Y_i \mid X_i \in \overline{A}, Y_i \in \overline{B}\}$ the vector space spanned by all products $XY,\ X \in \overline{A}, Y \in \overline{B}$, we have $AB \in \overline{A}\,\overline{B}$.

For

$$XY = (X[0]Y[0]), \begin{pmatrix} \rho(0,0)(XY) & \rho(0,1)(XY) \\ \rho(1,0)(XY) & \rho(1,1)(XY) \end{pmatrix} \in \overline{A}\,\overline{B}$$

with $X \in \overline{A}, Y \in \overline{B}$, the computation

$$
\begin{aligned}
\rho(0,0)(XY) &= (\rho(0,0)X)(\rho(0,0)Y) + (\rho(0,1)X)(\rho(1,0)Y) \\
\rho(0,1)(XY) &= (\rho(0,0)X)(\rho(0,1)Y) + (\rho(0,1)X)(\rho(1,1)Y) \\
\rho(1,0)(XY) &= (\rho(1,0)X)(\rho(0,0)Y) + (\rho(1,1)X)(\rho(1,0)Y) \\
\rho(1,1)(XY) &= (\rho(1,0)X)(\rho(0,1)Y) + (\rho(1,1)X)(\rho(1,1)Y)
\end{aligned}
$$

shows that $\overline{A}\,\overline{B}$ is recursively closed of dimension $\leq dim(\overline{A})\,dim(\overline{B})$. Assertion (iii) follows now from the obvious inclusion $\overline{AB} \subset \overline{A}\,\overline{B}$. $\qquad\square$

Remark 3.4. *Certain properties of binomial coefficients* (mod p^d) *are easy to study using "recurrence matrices" given by sequences of matrices of size $p^j \times p^j, j = 0, 1, \ldots$ with entries in the associative ring $\mathbb{Z}/p^d\mathbb{Z}$.*

3.2 Recursive presentations

An element $A \in \mathcal{A}$ is completely determined by the action of the shift maps $\rho(s,t)$ on its recursive closure $\overline{A}$ (spanned by $\{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^* A$), together with the restriction to $\overline{A}$ of the *augmentation map* $\pi_0 \in \mathcal{A}^*$ defined by projecting an element $X = (X[0], X[1], \ldots) \in \mathcal{A}$ onto its *initial value* $\pi_0(X[0], X[1], X[2], \ldots) = X[0] \in \mathbb{C}$.

A recurrence matrix A can thus be given by a finite amount of data: An (expression describing the) element A of the finite-dimensional vector space $\overline{A}$, the restriction (still denoted) $\pi_0 \in \overline{A}^*$ of the augmentation map expressing the initial values of elements in $\overline{A}$, and a 2×2 matrix $\rho = \begin{pmatrix} \rho(0,0) & \rho(0,1) \\ \rho(1,0) & \rho(1,1) \end{pmatrix} \in M_{2\times 2}(\overline{A} \otimes \overline{A}^*)$ of tensors encoding the shift maps. The coefficients of the matrix $A[n] \in M_{2^n \times 2^n}$ are then obtained by "contractions" of $\pi_0 \rho^n A$.

This leads to the notion of *recursive presentations*. A recursive presentation for $A \in \mathcal{R}$ is given by the choice of a basis $A_1 = A, \ldots, A_a$ of $\overline{A}$ (or more generally of a finite set $A_1, \ldots, A_a$ spanning a recursively closed vector space containing $\overline{A}$) and by recursive identities

$$A_j = \pi_0(A_j), \begin{pmatrix} \rho(0,0)A_j = \sum_{k=1}^a \rho(0,0)_{k,j} A_k & \rho(0,1)A_j = \sum_{k=1}^a \rho(0,1)_{k,j} A_k \\ \rho(1,0)A_j = \sum_{k=1}^a \rho(1,0)_{k,j} A_k & \rho(1,1)A_j = \sum_{k=1}^a \rho(1,1)_{k,j} A_k \end{pmatrix}$$

encoding the initial values $\pi_0(A_j)$ and the values $\rho(s,t)(A_j) \in \overline{A}$ of the shift maps. A recursive presentation defines the elements $A_1 = A, \ldots, A_a$ spanning (a recursively closed subspace containing) $\overline{A}$ recursively by expressing the four "blocks" of $A_j[n+1]$ as linear combinations of $A_1[n], \ldots, A_a[n]$.

Remark 3.5. *We use the convention that $0 \in \mathcal{R}$ admits the empty presentation and 0 is "the" element of an empty basis. We speak thus of "the" basis $A_1, \ldots$ of $\overline{0}$ representing $0 = A_1$.*

3.3 Saturation level

We denote by $\pi_l(A) = A[l]$ the projection of a matrix sequence $A = (A[0], A[1], A[2], \ldots) \in \mathcal{A}$ onto its square matrix $A[l]$ of size $2^l \times 2^l$. Similarly,

$$\pi_{\leq l}(A) = (\pi_0(A), \pi_1(A), \ldots, \pi_l(A)) = (A[0], A[1], \ldots, A[l]) \in \oplus_{j=0}^{l} M_{2^j \times 2^j}(\mathbb{C})$$

denotes the projection of the sequence A onto its first $l+1$ matrices.

The *saturation level* of a finite dimensional subspace $\mathcal{V} \subset \mathcal{A}$ is the smallest integer $N \in \mathbb{N}$ such that $\mathcal{K}_{\leq N}(\mathcal{V}) = \mathcal{K}_{\leq N+1}(\mathcal{V})$ where $\mathcal{K}_{\leq l}(\mathcal{V}) \subset \mathcal{V}$ is the kernel of the projection $\pi_{\leq l} : \mathcal{V} \longrightarrow \pi_{\leq l}(\mathcal{A}) = \oplus_{j=0}^{l} M_{2^j \times 2^j}$.

Proposition 3.6. *We have $\mathcal{K}_{\leq N}(\mathcal{V}) = \{0\}$ for the saturation level N of a finite-dimensional subspace $\mathcal{V} \subset \mathcal{A}$ which is recursively closed.*
In particular, $\pi_{\leq N} : \mathcal{V} \longrightarrow \oplus_{j=0}^{N} M_{2^j \times 2^j}$ defines an injection.

Proof The obvious inclusions $\rho(s,t)\mathcal{K}_{\leq l+1}(\mathcal{V}) \subset \mathcal{K}_{\leq l}(\mathcal{V})$ imply that $\mathcal{K}_{\leq N}(\mathcal{V}) = \mathcal{K}_{\leq N+1}(\mathcal{V}) \subset \mathcal{V}$ is recursively closed. Since the restriction to $\mathcal{K}_{\leq N+1}(\mathcal{V}) \subset \mathcal{K}_{\leq 0}$ of the augmentation map $\pi_0 : \mathcal{A} \longrightarrow \mathbb{C}$ is trivial, we have $(\mathbf{X}K)[0] = 0$ for all $\mathbf{X} \in \{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^*$ and for all $K \in \mathcal{K}_{\leq N}(\mathcal{V})$. This shows $\mathcal{K}_{\leq N}(\mathcal{V}) = \{0\}$. $\qquad\square$

Proposition 3.6 enables us to extract a basis from a finite set $\mathcal{S}$ spanning a recursively closed vector space $\mathcal{V} \subset \mathcal{R}$. Similarly, Proposition 3.6 allows the construction of a basis of the subspace $\overline{A} \subset \mathcal{V}$ for an element $A \in \mathcal{V}$ of a finite-dimensional recursively closed vector space $\mathcal{V} \subset \mathcal{R}$.

These operations are the necessary ingredients for effective computations in the algebra $\mathcal{R}$. Effectivity means that there exists an algorithm involving only a finite number of elementary operations in the groundfield $\mathbb{C}$ and a finite amount of data which computes the result of an algebraic expression (given by a non-commutative polynomial) involving (recursive presentations of) a finite number of elements in $\mathcal{R}$.

The necessary elementary algorithms can be briefly described as follows:

3.3.1 Multiplication of $A \in \mathcal{R}$ by a non-zero scalar $\lambda \in \mathbb{C}^*$

A presentation of λA is obtained from a presentation of A by multiplying the initial values $\pi(A_j) = A_j[0] \in \mathbb{C}$ with λ (and by keeping the same shift maps).

3.3.2 Addition of two elements $A, B \in \mathcal{R}$

Add a first element $A_1 + A_2$ having the obvious initial value $\pi_0(A + B) = A_1[0] + B_1[0]$ to the list of not necessarily linearly independent elements $A_1, \ldots, A_a, B_1, \ldots, B_b$ spanning $\overline{A + B}$. The elements $\rho(s, t)(A_1 + A_2), \rho(s, t)A_j, \rho(s, t)B_j$ are given by

$$\rho(s, t)(A_1 + B_1) = \sum_{k=1}^{a} \rho(s, t)_{k,1}^A A_k + \sum_{k=1}^{b} \rho(s, t)_{k,1}^B B_k$$

and

$$\rho(s, t)A_i = \sum_{k=1}^{a} \rho(s, t)_{k,i}^A A_k, \ \ \rho(s, t)B_j = \sum_{k=1}^{b} \rho(s, t)_{k,j}^B B_k$$

for all $0 \le s, t \le 1$, $1 \le i \le a$, $1 \le j \le b$ where $\rho(s, t)^A$ and $\rho(s, t)^B$ are the obvious shift maps with respect to bases $A = A_1, \ldots, A_a$ and $B = B_1, \ldots, B_b$ of $\overline{A}$ and $\overline{B}$. Working with the finite-dimensional recursively closed subspace $\mathbb{C}(A_1 + B_1) + \sum_{i=1}^{a} \mathbb{C}A_i + \sum_{j=1}^{b} \mathbb{C}B_j \subset \mathcal{R}$ one can now give a presentation of $A_1 + B_1$ by computing a basis of $\overline{A_1 + B_1}$, followed by the computation of the coefficients (with respect to this basis) of the shift maps.

Remark 3.7. *The algorithms 3.3.1 and 3.3.2 can be used to compare two elements $A, B \in \mathcal{R}$ by computing a presentation of $A - B$.*

3.3.3 Multiplication of two elements $A, B \in \mathcal{R}$

Consider the ab elements $C_{i,j} = A_i B_j$, $1 \le i \le a$, $1 \le j \le b$ with initial values $C_{i,j}[0] = \pi_0(A_i B_j) = A_i[0]B_j[0]$. Shift maps are given by

$$(*)\rho(s, t)C_{i,j} = \sum_{k=1}^{a} \sum_{l=1}^{b} \left(\rho(s, 0)_{k,i}^A \rho(0, t)_{l,j}^B + \rho(s, 1)_{k,i}^A \rho(1, t)_{l,j}^B \right) C_{k,l}$$

using the notations of 3.3.2. One constructs now a recursive presentation of $C_{1,1} = AB$ by proceeding as above using the recursively closed vector space $\sum_{i=1}^{a} \sum_{j=1}^{b} \mathbb{C}C_{i,j} \supset \overline{C_{1,1}}$.

Remark 3.8. *The formulae (*) occuring in 3.3.3 define an associative product on the set $\mathcal{L}$ of all equivalence-classes of finite-dimensional linear representations of the monoid $\{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^*$. Considering also direct sums of linear representations turns $\mathcal{L}$ into a semi-ring. The semi-ring $\mathcal{L}$ has a homomorphism φ into the semi-ring (with addition $\mathcal{V} + \mathcal{W} = \{X + Y \mid X \in \mathcal{V}, Y \in \mathcal{W}\}$ and product $\mathcal{V}\mathcal{W} = \{\sum_i X_i Y_i \mid X_i \in \mathcal{V}, Y_i \in \mathcal{W}\}$) formed by all finite-dimensional recursively closed subspaces of $\mathcal{R}$. The elements of $\varphi(L)$ have two equivalent descriptions: They can be identified with the subset $\mathcal{L}' \subset \mathcal{L}$ given by all equivalence classes of finite-dimensional linear representations of $\{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^*$ containing no equivalence class of*

a subrepresentation with multiplicity > 1. The second descriptions involves birecursively closed vector spaces which are defined as follows: A vector space $\mathcal{V} \subset \mathcal{R}$ is birecursively closed if it is recursively closed and if an arbitrary generic modification of the initial values in a recursive presentation of an element in $\mathcal{V}$ yields again a presentation of an element in $\mathcal{V}$.

The homomorphism of semi-rings φ associates to a finite-dimensional linear representation ρ_f of $\{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}$ the unique maximal birecursively closed subspace of $\mathcal{R}$ whoses shift maps involve only equivalence classes of subrepresentations in ρ_f.

Finite-dimensional birecursively closed subspaces of $\mathcal{R}$ are stable under direct sums and products and their semi-ring is the quotient semi-ring $\varphi(\mathcal{L})$ of $\mathcal{L}$.

3.4 The LU decomposition of a convergent non-singular element in $\mathcal{R}$

An element $P \in \mathcal{A}$ such that $P = \rho(0,0)P$ is called *convergent*. It is given by considering the sequence

$$P_{0,0}, \begin{pmatrix} P_{0,0} & P_{0,1} \\ P_{1,0} & P_{1,1} \end{pmatrix}, \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} \end{pmatrix}, \dots$$

of all square submatrices formed by the first 2^n rows and columns of an infinite "limit" matrix

$$\begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & \dots \\ P_{1,0} & P_{1,1} & P_{1,2} & \dots \\ \vdots & & & \end{pmatrix}.$$

Henceforth we denote generally a convergent element in $\mathcal{A}$ and the associated infinite matrix by the same letter. This should not lead to confusions except in cases where both interpretations are correct.

We call an infinite matrix P *non-singular* if the $k \times k$ square matrix $P(k)$ formed by its first k rows and columns has non-zero determinant for all $k \geq 1$. Such a non-singular matrix P has an $LU-$decomposition: It can be written as $P = LU$ with L lower triangular unipotent (1's on the diagonal) and U upper triangular non-singular. The identity $P = LU$ implies the equality $\det(P(k)) = \det(U(k))$ for all $k \geq 1$ and gives rise to an $LU-$decomposition in $\mathcal{A}$ by considering as above for $n = 0, 1, 2, \dots$ the submatrices formed by the first 2^n rows and columns of of P, L and U. If P is symmetric we have moreover $U = DL^t$ where D is diagonal non-singular and L^t is obtained by transposing the matrix L.

All proofs of the results presented in Section 2 boil down to $LU-$decompositions with $P = P^t = LU, L, D, U = DL^t \in \mathcal{R}$.

Remark 3.9. *Call an element $A \in \mathcal{A}$ non-singular if it involves only non-singular matrices $A[0], A[1], \ldots$. Such an element has an $LU-$decomposition (in the obvious sense) in $\mathcal{A}$.*

Proving the non-existence of an $LU-$decomposition in $\mathcal{R}$ for a suitable given non-singular recurrence matrix $A \in \mathcal{R}$ is probably difficult.

The related problem of constructing the (existing) recurrence matrices $L, U \in \mathcal{R}$ from the knowledge (of a recursive presentation) of $A = LU \in \mathcal{R}$ has however an algorithmic answer: One proceeds as for the existence of an inverse element by guessing recursive presentations for L and U using $LU-$decompositions of finitely many matrices $A[0], A[1], \ldots, A[N+1]$. In case of succes, the resulting hypothetical decomposition, if correct, can then be proven to hold. The necessary algorithm is obtained after minor modifications from the algorithm for computing the inverse $A^{-1} \in \mathcal{R}$ of an invertible element $A \in \mathcal{R}$ described in Section 4.1

4 Invertible recurrence matrices

The set of all recurrence matrices which are invertible in the algebra $\mathcal{R}$ forms the group of *units* in $\mathcal{R}$. Determining the inclusion in the unit group of $\mathcal{R}$ of a recurrence matrix A is perhaps a difficult problem without algorithmic solution. Indeed, we have the following result.

Proposition 4.1. *(i) For every natural integer n there exist invertible recurrence matrices $A, B = A^{-1} \in \mathcal{R}$ such that $\dim(\overline{A}) = 2$ and $\dim(\overline{B}) > n$.*

(ii) There exist elements in $\mathcal{R}$ which are invertible in the algebra $\mathcal{A}$ but not in the subalgebra $\mathcal{R}$ of recurrence matrices.

Remark 4.2. *The assumption $\dim(\overline{A}) = 2$ is optimal: Invertible recurrence matrices of complexity 1 form a subgroup (isomorphic to $\mathbb{C}^* \times GL_2(\mathbb{C})$) in $\mathcal{R}$.*

Proof of Proposition 4.1 For $\omega \in \mathbb{C}^*$, consider the convergent element

$$A = 1, \begin{pmatrix} 1 & \\ -\omega & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ -\omega & 1 & & \\ 0 & -\omega & 1 & \\ 0 & 0 & -\omega & 1 \end{pmatrix}, \cdots \in \mathcal{A}$$

consisting of lower triangular unipotent matrices with constant subdiagonal $-\omega$. It defines a recurrence matrix $A = A_1$ of complexity 2 recursively presented by

$$A_1 = 1, \begin{pmatrix} A_1 & 0 \\ A_2 & A_1 \end{pmatrix}, \quad A_2 = -\omega, \begin{pmatrix} 0 & A_2 \\ 0 & 0 \end{pmatrix}.$$

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Since A is given by a sequence of unipotent lower triangular matrices, it is invertible in the algebra $\mathcal{A}$ with inverse the convergent element

$$B = A^{-1} = 1, \begin{pmatrix} 1 & \\ \omega & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ \omega & 1 & & \\ \omega^2 & \omega & 1 & \\ \omega^3 & \omega^2 & \omega & 1 \end{pmatrix}, \cdots \in \mathcal{A}$$

whose limit is the infinite unipotent lower triangular Toeplitz matrix with constant subdiagonals associated to the geometric progression $1, \omega, \omega^2, \ldots$.

For k a strictly positive integer, we consider the element

$$S_k = \omega^{2^0 k}, \omega^{2^1 k} \begin{pmatrix} 1 & \omega \\ \omega & \omega^2 \end{pmatrix}, \omega^{2^2 k} \begin{pmatrix} 1 & \omega & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 & \omega^4 \\ \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ \omega^3 & \omega^4 & \omega^5 & \omega^6 \end{pmatrix}, \cdots \in \mathcal{A}$$

given by $(S_k[n])_{i,j} = \omega^{2^n k + i + j}$, $0 \le i, j < 2^n$. A straightforward computation shows $\rho(s,t) S_k = S_{s+t+2k}$ for all s, t. Since we have

$$B = 1, \begin{pmatrix} B & 0 \\ S_1 & B \end{pmatrix}$$

(using the notation of recursive presentations), the algebraic closure $\overline{B}$ of B is spanned by the set

$$\{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^* B = B \cup \{S_k \mid k \ge 1\} \ .$$

Since $\omega^N = 1$ implies $S_{k+N} = S_k$, this set contains at most $1 + N$ elements if ω is a root of 1 of finite order N.

In order to prove assertion (i), we consider the case where ω is a root of 1 having odd order $N > 2^n$ and we denote by $a \ge log_2(N+1) > n$ the order of 2 in the multiplicative group $(\mathbb{Z}/N\mathbb{Z})^*$. The action of the monoid $\rho(0,0)^{\mathbb{N}}$ on S_k corresponds then to the action of the Galois map $\omega \longmapsto \omega^2$ on the upper left coefficients

$$\omega^{2^0 k}, \omega^{2^1 k}, \omega^{2^2 k}, \omega^{2^3 k}, \ldots$$

of S_k. Elementary number theory (for instance reduction modulo 2 by choosing ω among the primitive N−th roots of 1 in the field extension $\mathbb{F}_{2^a}$ of degree a over $\mathbb{F}_2$) shows that the set $S_{2^{\mathbb{N}}}$ spans a subspace of dimension $a > n$ in $\overline{B}$. This implies assertion (i).

Consider now A as above with $\omega \in \mathbb{C} \setminus \overline{\mathbb{Q}}$ transcendental. This allows to consider ω as a variable and we have $\dim(\overline{B_\omega}) \ge \dim(\overline{B_\xi})$ for the complexity of the inverse element $B_\omega = A_\omega^{-1}$ where $\xi \in \overline{\mathbb{Q}}$ is any algebraic specialisation of ω. Assertion (ii) follows now from assertion (i). $\qquad\square$

Remark 4.3. *An example of $B = A^{-1} \in \mathcal{A} \setminus \mathcal{R}$ with $A \in \mathcal{R}$ invertible only in $\mathcal{A}$ is also given by A, B as above with $\omega = 2$ (the argument below works in fact for any ω of norm $|\omega| \geq 1$). Indeed, otherwise, up to a constant, the initial values $2, 2^2, 2^4, 2^8, \ldots$ of the sequence $\rho(1,0)B, \rho(0,0)\rho(1,0)B, \rho(0,0)^2\rho(1,0)B, \ldots$ should be bounded above by a geometric progression $1, \mu, \mu^2, \ldots$ for any positive μ exceeding the spectral radius (absolute value of the largest eigenvalue) of $\rho(0,0) \in \mathrm{End}(\overline{B})$.*

The simplest element of $\mathcal{R}$ with an inverse in $\mathcal{A} \setminus \mathcal{R}$ is perhaps given by the central diagonal recurrence matrix A of complexity 2 defined by $A[n] = (n+1)Id[n]$ where $Id[n]$ denotes the identity matrix of size $2^n \times 2^n$. We give two proofs that the inverse element A^{-1}, given by $A^{-1}[n] = \frac{1}{n+1}Id[n]$, has infinite complexity.

A first proof follows from the observation that the sequences of diagonal coefficients $\frac{1}{n+k+1} = (\rho(0,0)^k A^{-1})[n]$ form the rows of the Hilbert matrix of infinite rank with coefficients $H_{i,j} = \frac{1}{1+i+j}$, $0 \leq i, j$.

In order to give a second proof, we observe that all coefficients of A^{-1} are rational numbers and every prime number appears as a denominator in a suitable coefficient of A^{-1}. This is impossible for an element $B \in \mathcal{R}$ with rational coefficients. Indeed, such an element B has a recursive presentation with data (consisting of initial values and coefficients of shift maps with respect to a basis $\subset \{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^* B$ of $\overline{B}$) given by a finite set $\mathcal{D}$ of rational numbers. Since all coefficients of B are evaluations of integral polynomials on $\mathcal{D}$, all denominators of coefficients in B involve only prime numbers occuring in the denominators of the finite set $\mathcal{D}$.

Remark 4.4. *The quotient group of invertible elements modulo $\mathbb{C}^* Id$ can be turned into a metric group by considering the positive real function*

$$A \longmapsto \| A \| = \max(\log(\dim(\overline{A} + \mathbb{C}Id)), \log(\dim(\overline{A^{-1}} + \mathbb{C}Id)))$$

on the group Γ of units in $\mathcal{R}$. It satisfies $\| A \| \geq 0$ with $\| A \| = 0$ only for $A \in \mathbb{C}^* Id$, $\| AB \| \leq \| A \| + \| B \|$ and $\| A \| = \| A^{-1} \|$ and defines thus a left-invariant distance on the quotient group $\Gamma/\mathbb{C}^* Id$ by considering $d(A, B) = \| A^{-1} B \|$.

The corresponding group over a finite field has finitely many elements in balls of finite radii and it would be interesting to understand the generating function

$$\sum_{A \in \Gamma(\mathbb{F}_{p^e})} t^{\max(\dim(\overline{A}+\mathbb{C}Id), \dim(\overline{A^{-1}}+\mathbb{C}Id))} \in \mathbb{N}[[t]]$$

where the sum is over all elements of the unit group $\Gamma(\mathbb{F}_{p^e})$ of the algebra $\mathcal{R}(\mathbb{F}_{p^e})$ defined in the obvious way over the finite field $\mathbb{F}_{p^e}$.

The related generating function

$$\sum_{A \in \mathcal{R}(\mathbb{F}_{p^e})} t^{\dim(\overline{A})} \in \mathbb{N}[[t]]$$

counting all elements of given complexity in the algebra $\mathcal{R}(\mathbb{F}_{p^e})$ is probably fairly easy to compute. It has convergency radius 0 and is thus transcendental.

4.1 Algorithm for computing $A^{-1} \in \mathcal{R}$ for $A \in \mathcal{R}$ invertible in $\mathcal{R}$

The following algorithm computes the inverse of an element $A \in \mathcal{R}$ if it has an inverse in $\mathcal{R}$ and fails (does never stop and uses more and more memory) for an element A as in assertion (ii) of Proposition 4.1. Non-invertibility in $\mathcal{A}$ will eventually be detected (assuming exact arithmetics over the ground field) by exhibiting an integer k for which $A[k]$ is singular.

Input: A (presentation of a) recurrence matrix $A \in \mathcal{R}$.

Set $N = 1$ and $M = 1$.

Loop Compute, if possible, the matrices $B[0] = A[0]^{-1}, B[1] = A[1]^{-1}, \ldots, B[N + M + 1] = A[N + M + 1]^{-1}$.

If a matrix $A[k]$ with $k \leq N + M + 1$ is not invertible, print "The matrix $A[k]$ is singular and A has thus no inverse in $\mathcal{A}$" and stop.

We denote by $\tilde{B} \in \mathcal{A}$ the sequence $B[0], \ldots, B[N + M + 1]$ completed by arbitrary matrices $\tilde{B}[k]$ of size $2^k \times 2^k$ (for example by zero matrices) if $k > N + M + 2$.

For $n, m \in \mathbb{N}$ such that $n + m \leq N + M + 1$, we denote by $\mathcal{V}(n, m) \subset \oplus_{j=0}^{n} M_{2^j \times 2^j}(\mathbb{C})$ the vector-space spanned by all $1 + 4 + \cdots + 4^m$ elements $(\pi_{\leq n}(\mathbf{X}\tilde{B}))_{\mathbf{X} \in \mathcal{X}^{\leq m}}$ where $\mathcal{X}^{\leq m}$ denotes the set of all $\frac{1 - 4^{m+1}}{1 - 4} = 1 + 4 + 4^2 + \cdots + 4^m$ words of length $\leq m$ in the alphabet $\mathcal{X} = \{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}$.

If $\dim(\mathcal{V}(N, M)) < \dim(\mathcal{V}(N + 1, M))$, then increase N by 1 and iterate the Loop.

If $\dim(\mathcal{V}(N, M)) < \dim(\mathcal{V}(N, M + 1))$, then increase M by 1 and iterate the Loop.

Setting $d = \dim(\mathcal{V}(N, M))$, the identity $\dim(\mathcal{V}(N, M)) = \dim(\mathcal{V}(N + 1, M))$ shows that the finite sequence $B[0], \ldots, B[N + M + 1]$ can be completed to a uniquely defined element $B \in \mathcal{R}$ which is of complexity d and of saturation level $\leq N$. Use the natural isomorphisms between $\mathcal{V}(N, M), \mathcal{V}(N + 1, M)$ and $\mathcal{V}(N, M+1)$ and the inclusions $\rho(s, t)\mathcal{V}(N, M+1) \subset \mathcal{V}(N, M)$, $0 \leq s, t \leq 1$ (where $\rho(s, t)$ acts in the obvious way) for writing down a recursive presentation of B.

Check if $AB = 1$ using the algorithms of Section 3.3.

If yes, print the presentation found for B and stop.

Otherwise, increase N by 1 and iterate the Loop.

End of Loop

Remark 4.5. *The above algorithm can perhaps be improved. In particular, it is probably not necessary to consider all $(4^{M+1} - 1)/3$ words of $\mathcal{X}^{\leq M} \subset \mathcal{X}^*$ during the computation of the dimension of $\mathcal{V}(N + 1, M)$.*

5 Modulo 2 and $2-$valuations

Proof of Theorem 2.1 The infinite symmetric Pascal matrix P with co-efficients $\left(\binom{i+j}{i} \pmod 2 \right) \in \{0,1\}$ for $0 \le i,j$ defines a convergent element (still denoted) $P \in \mathcal{A}$. It follows from Lucas's formula (see Section 2.1) that P is a recurrence matrix of complexity 1 recursively presented by

$$P = 1, \begin{pmatrix} P & P \\ P & \end{pmatrix}$$

(zero-entries are omitted). The recurrence matrix $P \in \mathcal{R}$ has an LU decomposition in $\mathcal{R}$ given by the equality $P = LDL^t$ with $L, D \in \mathcal{R}$ of complexity 1 defined by the recursive presentations

$$L = 1, \begin{pmatrix} L & \\ L & L \end{pmatrix} \text{ and } D = 1, \begin{pmatrix} D & \\ & -D \end{pmatrix}$$

where L is lower triangular unipotent and D is diagonal. An easy analysis of the coefficients of the diagonal recurrence matrix D ends the proof. □

Remark 5.1. *The convergent lower triangular recurrence matrix L and the convergent diagonal recurrence matrix D correspond to the infinite limit-matrices (still denoted) L, D with coefficients given by $L_{i,j} = \left(\binom{i+j}{i} \pmod 2 \right) \in \{0,1\}$ (for $0 \le i,j$) and $D_{n,n} = (-1)^{\nu_0 + \nu_1 + \nu_2 + \cdots} = (-1)^{ds(n)}$ where $n = \sum_{j \ge 0} \nu_j 2^j \ge 0$ is a binary integer.*

Remark 5.2. *Recurrence matrices of complexity 1 are, up to scalars, of the form $1, M, M \otimes M, M \otimes M \otimes M, \ldots$ where M is a complex 2×2 matrix.*

It follows that the matrices L and D (and thus also $P = LDL^t$) involved in the proof of Theorem 2.1 are invertible in $\mathcal{R}$. The recurrence matrix D is its own inverse. The inverse L^{-1} of L is recursively presented by $L^{-1} = 1, \begin{pmatrix} L^{-1} & \\ -L^{-1} & L^{-1} \end{pmatrix}$.

Remark 5.3. *The spectrum of recurrence matrices of complexity 1 is easy to compute: Given a square matrix M of size $d \times d$ with characteristic polynomial $\prod_{j=1}^{d}(t - \lambda_j)$ we have*

$$\prod_{(j_1, \ldots, j_n) \in \{1, \ldots, d\}^n} (t - \lambda_{j_1} \cdots \lambda_{j_n})$$

for the characteristic polynomial of the iterated tensor power $M^{\otimes^n}$.

Proof of Theorem 2.3 The infinite matrix V with coefficients

$$V_{s,t} = i^{v_2\left(\binom{s+t}{s}\right)} = i^{ds(s) + ds(t) - ds(s+t)}$$

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gives rise to a convergent element $V \in \mathcal{A}$. A bit of work using Kummer's formulae (see Section 2.2) shows that $V = V_1$ is recurrence matrix given by the recursive presentation

$$V_1 = 1, \begin{pmatrix} V_1 & V_2 \\ V_2 & iV_1 \end{pmatrix}$$

$$V_2 = 1, \begin{pmatrix} V_1 & -iV_1 + (1+i)V_2 \\ -iV_1 + (1+i)V_2 & -V_1 \end{pmatrix}$$

We have $V = LDL^t \in \mathcal{R}$ with $L = L_1 \in \mathcal{R}$ recursively presented by

$$L_1 = 1, \begin{pmatrix} L_1 & \\ L_3 & L_4 \end{pmatrix}$$

$$L_2 = 0, \begin{pmatrix} 0 & -iL_2 \\ -L_1 + L_3 & -iL_2 - iL_4 \end{pmatrix}$$

$$L_3 = 1, \begin{pmatrix} L_1 & L_2 \\ -iL_1 + (1+i)L_3 & L_2 + (1+i)L_4 \end{pmatrix}$$

$$L_4 = 1, \begin{pmatrix} L_1 & \\ (1-i)L_1 + iL_3 & L_4 \end{pmatrix}$$

and with diagonal $D = D_1 \in \mathcal{R}$ recursively presented by

$$D_1 = 1, \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}$$

$$D_2 = -1 + i, \begin{pmatrix} D_3 & \\ & 2D_1 - D_2 + 2D_3 \end{pmatrix}$$

$$D_3 = -1 + i, \begin{pmatrix} D_3 & \\ & -D_2 \end{pmatrix}$$

An analysis (left to the reader) of the diagonal entries of D_1 ends the proof.
□

Remark 5.4. *The recurrence matrices L, D and $V = LDL^t$ are invertible in $\mathcal{R}$, see [3].*

6 Beeblebrox reduction

This section is devoted to proofs and complements involving the Beeblerox reduction $\chi_B(\binom{n}{k})$ of binomial coefficients.
Proof of Theorem 2.5 We have

$$\binom{2n}{2k} = \frac{(2n)\cdots(2n-2k+1)}{(2k)\cdots 1} = \binom{n}{k}\frac{(2n-1)(2n-3)\cdots(2n-2k+1)}{(2k-1)(2k-3)\cdots 1}$$

where both the numerator and the denominator of the fraction

$$F = \frac{(2n-1)(2n-3)\cdots(2n-2k+1)}{(2k-1)(2k-3)\cdots 1}$$

contain k terms. If k is even, we have $F \equiv 1 \pmod 4$ since the numerator and denominator of the fraction F contain both $k/2$ factors $\equiv 1 \pmod 4$ and $k/2$ factors $\equiv -1 \pmod 4$. If k and n are both odd, the numerator and denominator of F contain both $(k+1)/2$ factors $\equiv 1 \pmod 4$ and $(k-1)/2$ factors $\equiv -1 \pmod 4$ and we have again $F \equiv 1 \pmod 4$. If k is odd and n is even, then both binomial coefficients $\binom{2n}{2k}$ and $\binom{n}{k}$ are even and we have thus $\chi_B(\binom{2n}{2k}) = \chi_B(\binom{n}{k}) = 0$. This proves the first equality.

The binomial coefficient $\binom{2n}{2k+1}$ is obviously even and this implies the second equality.

In the next case we have

$$\binom{2n+1}{2k} = \binom{n}{k} \frac{(2n+1)(2n-1)\cdots(2n-2k+3)}{(2k-1)(2k-3)\cdots(1)}$$

and the last fraction equals $1 \pmod 4$ if k is even.

For k odd and n even, we have $\chi_B(\binom{2n+1}{2k}) = \chi_B(\binom{n}{k}) = 0$ since $\binom{2n+1}{2k} \equiv \binom{n}{k} \equiv 0 \pmod 2$.

For n, k both odd, the correction $(-1)^k = -1$ equals the fraction modulo 4. This ends the proof of the third equality.

In the case of the last equality, we have

$$\binom{2n+1}{2k+1} = \frac{2n+1}{2k+1}\binom{2n}{2k}$$

which is even if $n \equiv 0 \pmod 2$ and $k \equiv 1 \pmod 2$. If $n \equiv k \pmod 2$ then $\frac{2n+1}{2k+1} \equiv 1 \pmod 4$. For n odd and k even we have $\frac{2n+1}{2k+1} \equiv -1 = (-1)^{n(k+1)} \pmod 4$. The first equality and these observations complete the proof. $\square$

6.1 Proof of Theorem 2.6

Proof As in Section 3.4, we consider the element (still denoted) $Z \in \mathcal{A}$ associated to the infinite matrix Z with coefficients $Z_{s,t} = \chi_B(\binom{s+t}{s})$, $0 \le s, t$.

We have to show that Z is a recurrence matrix and we have to find a recursive presentation for Z. This can be done in the following way: We consider *left shift-maps* $\lambda(0,0), \lambda(0,1), \lambda(1,0), \lambda(1,1)$ which associate to $A = (A[0], A[1], \dots) \in \mathcal{A}$ the element $\lambda(s,t)A \in \mathcal{A}$ where $(\lambda(s,t)A)[n]$ is the submatrix of $A[n+1]$ corresponding to row-indices $\equiv s \pmod 2$ and column-indices $\equiv t \pmod 2$. A subspace $\mathcal{V} \subset \mathcal{A}$ is *left-recursively closed* if it is invariant under all four left shift-maps and the *left-recursive closure* $\overline{A}^\lambda \subset \mathcal{A}$ of $A \in \mathcal{A}$ is the smallest left-recursively closed subspace containing A. Theorem 2.5 implies that $\overline{Z}^\lambda$ is finite-dimensional. One shows that $dim(\overline{A}^\lambda) = dim(\overline{A})$ for all $A \in \mathcal{A}$ (see for example [4] for the details). This proves that Z is a recurrence matrix. A little bit of work based on

properties of the saturation index shows now that $Z = Z_1$ is given by the recursive presentation

$$Z_1 = 1, \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & 0 \end{pmatrix}$$
$$Z_2 = 1, \begin{pmatrix} Z_1 & Z_2 \\ -Z_3 & 0 \end{pmatrix}$$
$$Z_3 = 1, \begin{pmatrix} Z_1 & -Z_2 \\ Z_3 & 0 \end{pmatrix}$$

We have the identity $Z = LDL^t$ with $L = L_1 \in \mathcal{R}$ lower triangular unipotent given by the recursive presentation

$$L_1 = 1, \begin{pmatrix} L_1 & \\ L_3 & L_4 \end{pmatrix} \qquad L_2 = 2, \begin{pmatrix} -\frac{2}{3}L_1 & 2L_2 \\ \frac{2}{3}L_3 & 2L_4 \end{pmatrix}$$
$$L_3 = 1, \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \qquad L_4 = 1, \begin{pmatrix} L_1 & \\ \frac{1}{3}L_3 & L_4 \end{pmatrix}$$

The diagonal matrix $D = D_1 \in \mathcal{R}$ has recursive presentation

$$D_1 = 1, \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}, D_2 = -1, \begin{pmatrix} 3D_1 & \\ & \frac{1}{3}D_2 \end{pmatrix}$$

An easy inspection of the diagonal entries of D_1 completes the proof. $\qquad \square$

Remark 6.1. *The birecursively closed subspaces of $\mathcal{A}$ appearing in Remark 3.8 are the recursively closed subspaces of $\mathcal{A}$ which are also left-recursively closed, ie. invariant under all four left shift-maps mentionned above.*

6.1.1 The group Γ_L

All four recurrence matrices L_1, L_2, L_3, L_4 involved in our recursive presentation of $L = L_1$ are invertible in $\mathcal{R}$. Their inverses are given by

$$L_1^{-1} = M_1, \ L_2^{-1} = -\frac{1}{2}M_3, \ L_3^{-1} = -\frac{1}{2}M_2, \ L_4^{-1} = M_4$$

with M_1, M_2, M_3, M_4 recursively presented by

$$M_1 = 1, \begin{pmatrix} M_1 & \\ M_3 & M_4 \end{pmatrix} \qquad M_2 = -2, \begin{pmatrix} 2M_1 & 2M_2 \\ 2M_3 & 2M_4 \end{pmatrix}$$
$$M_3 = -1, \begin{pmatrix} M_1 & M_2 \\ \frac{1}{3}M_3 & -\frac{1}{3}M_4 \end{pmatrix} \qquad M_4 = 1, \begin{pmatrix} M_1 & \\ \frac{1}{3}M_3 & M_4 \end{pmatrix}$$

We have the curious inclusions

$$\rho(0,0)L \in \mathbb{Q}L_1, \qquad \rho(0,1)L \in \mathbb{Q}L_2,$$
$$\rho(1,0)L \in \mathbb{Q}L_3, \qquad \rho(1,1)L \in \mathbb{Q}L_4$$

for $L \in \overline{L_1} = \overline{L_2} = \overline{L_3} = \overline{L_4} = \oplus_{j=1}^4 \mathbb{C}L_j$. The analogous property holds also for their inverses

$$M_1 = L_1^{-1}, \ M_2 = -2L_3^{-1}, \ M_3 = -2L_2^{-1}, \ M_4 = L_4^{-1} \ .$$

This suggests that it would perhaps be interesting to understand the group $\Gamma_L = \langle a, b, c, d \rangle \subset \mathcal{R}$ generated by the normalised recurrence matrices

$$a = L_1 = 1, \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 1 & \frac{1}{3} & 1 \end{pmatrix}, \ldots$$

$$b = \frac{1}{2}L_2 = 1, \begin{pmatrix} -\frac{1}{3} & 2 \\ \frac{1}{3} & 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} & 0 & -\frac{2}{3} & 4 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 2 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}, \ldots$$

$$c = L_3 = 1, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -\frac{2}{3} & 4 \\ 1 & 1 & \frac{2}{3} & 2 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & \frac{1}{3} & 1 \end{pmatrix}, \ldots$$

$$d = L_4 = 1, \begin{pmatrix} 1 & \\ \frac{1}{3} & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \frac{1}{3} & \frac{2}{3} & 1 & \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}, \ldots$$

corresponding to the invertible "projective" elements $\mathbb{Q}^* L_i$. In particular, it would be interesting to understand if Γ_L is a linear group. This would certainly be implied by the existence (which I ignore) of a natural integer N for which the projection $\Gamma_L \longrightarrow \pi_{\leq N}(\Gamma_L)$ is one-to-one.

L. Bartholdi communicated to me the following list implying all relations of length ≤ 12 in Γ_L:

$$bd^{-1}ca^{-1},$$
$$ca^{-1}ca^{-1},$$
$$cad^{-1}a^{-1}d^2a^{-1}b^{-1},$$
$$cad^{-1}c^{-1}bda^{-1}b^{-1},$$
$$cda^{-2}dad^{-1}b^{-1},$$
$$ca^2d^{-1}a^{-2}dada^{-2}b^{-1},$$
$$d^2ad^{-2}b^{-1}cada^{-3},$$
$$cdad^{-2}a^{-1}dada^{-2}b^{-1}.$$

He observed that they are all of the form $u_n v_n^{-1} w_n x_n^{-1}$, where u_n, v_n, w_n, x_n are positive words of length n with respect to the generators $\{a, b, c, d\}$.

More generally, it should also be interesting to understand the subalgebra $\mathcal{L} \subset \mathcal{R}$ generated by the recurrence matrices $a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1} \in \mathcal{R}$.

The algebra $\mathcal{L}$ is of course a quotient of the group algebra $\mathbb{C}[\Gamma_L]$ and it would be interesting to describe the kernel of the associated homomorphism.

The algebra $\mathcal{L}$ is a quotient of the free non-commutative algebra

$$\mathcal{N} = \mathbb{C}\langle A, A^{-1}, B, B^{-1}, C, C^{-1}, D, D^{-1}\rangle$$

in eight free non-commutative variables $A^{\pm 1}, B^{\pm 1}, C^{\pm 1}, D^{\pm 1}$. The corresponding natural homomorphism $\pi : \mathcal{N} \longrightarrow \mathcal{L}$ (given by $Z^{\pm 1} \longmapsto z^{\pm 1} \in \mathcal{R}$ for $(Z, z) \in \{(A, a), (B, b)(C, c), (D, d)\}$) factorises through the group algebra of the abstract group Γ_L. The kernel $\mathcal{I} = \ker(\pi) \subset \mathcal{N}$ contains thus all relation of Γ_L and in particular the trivial relations

$$AA^{-1} - 1, \ BB^{-1} - 1, \ CC^{-1} - 1, \ DD^{-1} - 1 \ .$$

In order to gain some information on $\mathcal{I}$, we can consider the morphism of algebras $\mu_1 : \mathcal{N} \longrightarrow M_{2\times 2}(\mathcal{N})$ given by

$$A \longmapsto \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, \qquad A^{-1} \longmapsto \begin{pmatrix} A^{-1} & 0 \\ -B^{-1} & D^{-1} \end{pmatrix},$$
$$B \longmapsto \begin{pmatrix} -A/3 & 2B \\ C/3 & D \end{pmatrix}, \quad B^{-1} \longmapsto \begin{pmatrix} -A^{-1} & 2C^{-1} \\ B^{-1}/3 & D^{-1}/3 \end{pmatrix},$$
$$C \longmapsto \begin{pmatrix} A & 2B \\ C & D \end{pmatrix}, \qquad C^{-1} \longmapsto \begin{pmatrix} -A^{-1} & 2C^{-1} \\ B^{-1} & -D^{-1} \end{pmatrix},$$
$$D \longmapsto \begin{pmatrix} A & 0 \\ C/3 & D \end{pmatrix}, \qquad D^{-1} \longmapsto \begin{pmatrix} A^{-1} & 0 \\ -B^{-1}/3 & D^{-1} \end{pmatrix},$$

This morphism factors through π and induces a homomorphism of algebras $\overline{\mu}_1 : \mathcal{L} \longrightarrow M_{2\times 2}(\mathcal{L})$ which removes simply the first matrix $X[0]$ from an element $X[0], X[1], \cdots \in \mathcal{L}$. This is due to the definition of μ_1 which corresponds to the maps $\mathcal{R} \longrightarrow M_{2\times 2}(\mathcal{R})$ given by

$$X \longmapsto \begin{pmatrix} \rho(0,0)X & \rho(0,1)X \\ \rho(1,0)X & \rho(1,1)X \end{pmatrix}$$

for $X \in \{a^{\pm 1}, b^{\pm -1}, c^{\pm 1}, d^{\pm 1}\}$. We have thus in particular $\mu_1(\mathcal{I}) \subset M_{2\times 2}(\mathcal{I})$. Application of μ_1 to some known element in $\mathcal{I}$ can sometimes be used for the discovery of new elements in $\mathcal{I}$: The computation

$$\mu_1(AA^{-1}) = \begin{pmatrix} A & \\ C & D \end{pmatrix}\begin{pmatrix} A^{-1} & \\ -B^{-1} & D^{-1} \end{pmatrix} = \begin{pmatrix} AA^{-1} & \\ CA^{-1} - DB^{-1} & DD^{-1} \end{pmatrix}$$

shows that application of μ_1 to the trivial relation $AA^{-1} - 1 \in \mathcal{I}$ implies the already known inclusions $AA^{-1} - 1, DD^{-1} - 1 \in \mathcal{I}$ and produces the non-trivial relation $CA^{-1} - DB^{-1} \in \mathcal{I}$. Since elements of $\mathcal{I}$ involving only two monomials induce relations on the group Γ_L, we get the relation $bd^{-1}ca^{-1}$ in Γ_L which is the first relation in Bartholdi's list.

"Iterating" the map μ_1 produces homomorphisms $\mu_n : \mathcal{N} \longrightarrow M_{2^n \times 2^n}(\mathcal{N})$ with similar properties. In particular, we have $\mu_n(\mathcal{I}) \subset M_{2^n \times 2^n}(\mathcal{I})$. Indexing the coefficients of matrices in $M_{2^n \times 2^n}(\mathcal{N})$ by elements $\mathbf{X} \in \{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^n$, we get linear maps $\mu_{\mathbf{X}} : \mathcal{N} \longrightarrow \mathcal{N}$ by considering the coefficient corresponding to $\mathbf{X}$ in $\mu_n(\mathcal{N})$. The reader should be warned that the map $\mathbf{X} \longmapsto \mu_{\mathbf{X}} \in End(\mathcal{N})$ is not a morphism of monoids from $\{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^*$ into $End(\mathcal{N})$. The monoid generated by all maps $\mu_{\mathbf{X}}, \mathbf{X} \in \{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^*$, preserves however the ideal $\mathcal{I}$.

Denoting by $\mathcal{N}(\mathbb{Z})$ the subring of noncommutative polynomials with integral coefficients, relations of the form $r_1 = r_2$ in Γ_L are in bijection with pairs of roots in the infinite-dimensional Euclidean lattice (with respect to the orthonormal basis given by monomials) $\mathcal{I} \cap \mathcal{N}(\mathbb{Z})$.

The peculiar form of all relations in Bartholdi's list is partially explained by the formulae for μ_1. They imply that the maps $\mu_{\mathbf{X}}$ preserve sign structures: The vector space spanned by the orbit under the monoid generated by the maps $\mu_{\mathbf{X}}$ of a relation of the form $u_n v_n^{-1} = x_n w_n^{-1}$ contains only relations of the same form.

It would be interesting to know if the ideal $\mathcal{I}$ is finitely generated as an $\{\mu_{\mathbf{X}}\}_{\mathbf{X} \in \{X_{0,0}, X_{0,1}, X_{1,0}, X_{1,1}\}^*}$—module: Otherwise stated, does $\mathcal{I}$ contain a finite subset $\mathcal{G}$ such that $\mathcal{I}$ is the smallest bilateral ideal which contains $\mathcal{G}$ and which is preserved by all maps $\mu_{\mathbf{X}}$?

Remark 6.2. *The techniques used in this Section can of course be applied to other subsets in $\mathcal{R}$. One needs a (preferably finitely generated) algebra $\mathcal{S}$ (eg. the algebra generated by suitable elements of a subgroup in $\mathcal{R}$) such that the recursive closure of every element in $\mathcal{S}$ is spanned by elements of $\mathcal{S}$. The choice of a generating set $\mathcal{G}$ allows to consider the free non-commutative algebra $\mathcal{N}$ on $\mathcal{G}$ which gives rise to the natural surjective homomorphism $\pi : \mathcal{N} \longrightarrow \mathcal{S}$. Choosing lifts of the shift maps, one constructs homomorphisms $\mu_{\mathbb{N}} : \mathcal{N} \longrightarrow M_{2^n \times 2^n}(\mathcal{N})$ giving rise to the linear maps $\mu_{\mathbf{X}} \in End(\mathcal{N})$ preserving the bilateral ideal $\mathcal{I} = \ker(\pi)$.*

The maps $\mu_{\mathbf{X}}$ can be choosen in order to preserve the grading of $\mathcal{N}$ if the generating set $\mathcal{G}$ spans a recursively closed subspace.

6.1.2 The inverses of D_1, D_2 and the group Γ_Z

The inverses of the diagonal recurrence matrices D_1, D_2 (defined by the decomposition $Z = LD_1L^t$ and by $D_2 = \rho(1,1)D_1$) are $E_1 = D_1^{-1}$, $E_2 = D_2^{-1}$ recursively presented by

$$E_1 = 1, \begin{pmatrix} E_1 & \\ & E_2 \end{pmatrix}, E_2 = -1, \begin{pmatrix} \frac{1}{3}E_1 & \\ & 3E_2 \end{pmatrix}$$

The inverses of the matrices Z_1, Z_2, Z_3 are $U_1 = Z_1^{-1}, U_3 = Z_2^{-1}, U_2 =$

Z_3^{-1} recursively presented by

$$U_1 = 1, \begin{pmatrix} 0 & U_2 \\ U_3 & U_1 - U_2 - U_3 \end{pmatrix} ,$$
$$U_2 = 1, \begin{pmatrix} 0 & U_2 \\ -U_3 & -U_1 + U_2 + U_3 \end{pmatrix} ,$$
$$U_3 = 1, \begin{pmatrix} 0 & -U_2 \\ U_3 & -U_1 + U_2 + U_3 \end{pmatrix} .$$

Since $Z_1 = Z_1^t$ and $Z_2 = Z_3^t$, the group $\Gamma_Z = \langle a, b, c \rangle \subset \mathcal{R}$ generated by $a = Z_1, b = Z_2, c = Z_3$ has an involutive automorphism given by $a \mapsto a^{-1}$, $b \mapsto c^{-1}$, $c \mapsto b^{-1}$. Two relations in Γ_Z are $(ab^{-1})^2$ and $ab = ca$.

Using the relation $a^2 = cb$ following from the computation

$$a^2 = aab^{-1}b = aba^{-1}b = caa^{-1}b = cb ,$$

every element of Γ_Z can be expressed as an element of $a^\epsilon \langle b, c \rangle$ with $\epsilon \in \{0, 1\}$.

Since $\det(\pi_2(a)) = -1$ and $\det(\pi_2(b)) = \det(\pi_2(c)) = 1$, the subgroup generated by b, c is of index 2 in Γ_Z.

6.2 The triangular Beeblebrox matrix

We define the lower triangular Beeblebrox matrix as the infinite lower triangular matrix with coefficients $L_{s,t} = \chi_B(\binom{s}{t}), 0 \le s, t$, given by the Beeblebrox reduction of binomial coefficients.

One of the main results of [7] states that any fixed row of L contains either no coefficients -1 or the same number (given by a power of 2) of coefficients 1 and -1. This can of course also be deduced from Theorem 2.5 or by computing LJ where J is the "recurrence vector" obtained by considering the sequence of column vectors

$$(1), \ (1,1)^t, \ (1,1,1,1)^t, \ (1,1,1,1,1,1,1,1)^t, \dots .$$

The triangular Beeblebrox matrix L defines a recurrence matrix (still denoted) $L = L_1 \in \mathcal{R}$ recursively presented by

$$L_1 = 1, \begin{pmatrix} L_1 & \\ L_2 & L_3 \end{pmatrix}, L_2 = 1, \begin{pmatrix} L_1 & \\ L_2 & -L_3 \end{pmatrix}, L_3 = 1, \begin{pmatrix} L_1 & \\ -L_2 & L_3 \end{pmatrix} .$$

6.2.1 The recurrence matrices L_i^{-1}

The lower triangular recurrence matrices L_1, L_2, L_3 defined above are invertible in $\mathcal{R}$ with inverse elements $M_1 = L_1^{-1}, M_2 = L_2^{-1}, M_3 = L_3^{-1}$ recursively presented by

$$M_1 = 1, \begin{pmatrix} M_1 & \\ M & M_3 \end{pmatrix}, M_2 = 1, \begin{pmatrix} M_1 & \\ -M & -M_3 \end{pmatrix}, M_3 = 1, \begin{pmatrix} M_1 & 0 \\ -M & M_3 \end{pmatrix}$$

where $M = M_1 - M_2 - M_3$.

Proposition 6.3. *The map*

$$L_1 \longmapsto \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, L_2 \longmapsto \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, L_3 \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

(where the three matrices correspond to the three affine maps $(x, y) \mapsto (x + 2, y), (x, y) \mapsto (y+1, x+1), (x, y) \mapsto (x, y+2)$ of $\mathbb{R}^2$) defines a faithful linear representation of the group $\langle L_1, L_2, L_3 \rangle \subset \mathcal{R}$ generated by L_1, L_2, L_3.

Moreover, the group homomorphism $L_i \longmapsto \pi_2(L_i)$ is faithful on $\langle L_1, L_2, L_3 \rangle$.

Proof We check that L_1 and L_3 commute. They generate thus an abelian subgroup Γ_a which is easily seen to be free abelian of rank 2 by considering the 4×4 matrices $\pi_2(L_1)$ and $\pi_2(L_3)$. Checking the relations

$$L_2^2 = L_1 L_3, \ L_2 L_3 = L_1 L_2, \ L_2 L_1 = L_3 L_2$$

shows that Γ_a is of index 2 in $\langle L_1, L_2, L_3 \rangle$ and these relations define the affine group of Proposition 6.3.

Faithfulness of the homomorphism $L_i \longmapsto \pi_2(L_i)$ follows from the observation that the subgroup generated by $\pi_2(L_1), \pi_2(L_3)$ is free abelian of rank 2 and does not contain $\pi_2(L_2)$. $\qquad\square$

7   On the Jacobi-Dirichlet character

This section contains the most important data for proving Theorem 2.7. We omit the somewhat lengthy details.

Tedious work proving formulae analogous to Theorem 2.5 or general principles show that the matrix $J = J_1$ with coefficients $\chi_J(\binom{s+t}{s})$, $0 \le s, t$ is of complexity 9 and has a recursive presentation given by

$$J_1 = 1, \begin{pmatrix} J_1 & J_2 \\ J_2^t & \end{pmatrix},$$

$$J_2 = 1, \begin{pmatrix} J_3 & J_4 \\ J_5 & \end{pmatrix}, \qquad J_2^t = 1, \begin{pmatrix} J_3^t & J_5^t \\ J_4^t & \end{pmatrix},$$

$$J_3 = 1, \begin{pmatrix} J_1 & J_2 \\ -J_2^t & \end{pmatrix}, \qquad J_3^t = 1, \begin{pmatrix} J_1 & -J_2 \\ J_2^t & \end{pmatrix},$$

$$J_4 = 1, \begin{pmatrix} J_3 & J_4 \\ -J_5 & \end{pmatrix}, \qquad J_4^t = 1, \begin{pmatrix} J_3^t & -J_5^t \\ J_4^t & \end{pmatrix},$$

$$J_5 = -1, \begin{pmatrix} J_3^t & J_5^t \\ -J_4^t & \end{pmatrix}, \qquad J_5^t = -1, \begin{pmatrix} J_3 & -J_4 \\ J_5 & \end{pmatrix},$$

where $X^t = X[0]^t, X[1]^t, X[2]^t, \ldots$ for $X \in \mathcal{A}$.

The matrix J has an $J = LDL^t$ decomposition in $\mathcal{R}$ with L of complexity 20 and D of complexity 4.

The lower triangular recurrence matrix $L = L_1$ involved in the decomposition $J = LDL^t$ has the recursive presentation:

$$L_1 = 1, \begin{pmatrix} L_1 & 0 \\ L_2 & L_3 \end{pmatrix}, \qquad L_2 = 1, \begin{pmatrix} L_4 & L_5 \\ L_6 & L_7 \end{pmatrix},$$

$$L_3 = 1, \begin{pmatrix} L_8 & 0 \\ L_9 & L_{10} \end{pmatrix}, \qquad L_4 = 1, \begin{pmatrix} L_1 & L_{11} \\ L_2 & L_3 \end{pmatrix},$$

$$L_5 = 2, \begin{pmatrix} L_{12} & L_{13} \\ 0 & 0 \end{pmatrix}, \qquad L_6 = 1, \begin{pmatrix} L_4 & L_{14} \\ L_6 & L_7 \end{pmatrix},$$

$$L_7 = 1, \begin{pmatrix} L_8 - L_{12} & L_{15} \\ L_9 & L_{10} \end{pmatrix}, \qquad L_8 = 1, \begin{pmatrix} L_1 & 0 \\ 1/3L_2 & L_3 \end{pmatrix},$$

$$L_9 = 1/3, \begin{pmatrix} L_4 & 3L_5 \\ 1/3L_6 & L_7 \end{pmatrix}, \qquad L_{10} = 1, \begin{pmatrix} L_8 & 0 \\ 1/3L_9 & L_{10} \end{pmatrix},$$

$$L_{11} = 2, \begin{pmatrix} L_{16} & L_{13} \\ L_{17} & L_{18} \end{pmatrix}, \qquad L_{12} = 4/3, \begin{pmatrix} L_{19} & 4L_5 \\ 0 & 0 \end{pmatrix},$$

$$L_{13} = 4, \begin{pmatrix} 2/3L_{12} & 2L_{13} \\ 0 & 0 \end{pmatrix}, \qquad L_{14} = 0, \begin{pmatrix} L_{12} + L_{16} & L_{20} \\ L_{17} & L_{18} \end{pmatrix},$$

$$L_{15} = 2, \begin{pmatrix} -2/3L_{12} + L_{16} & L_{13} \\ 1/3L_{17} & L_{18} \end{pmatrix}, \quad L_{16} = -2/3, \begin{pmatrix} -2/3L_1 & 2L_{11} \\ 2/3L_2 & 2L_3 \end{pmatrix},$$

$$L_{17} = 2/3, \begin{pmatrix} -2/3L_4 & -4L_5 + 2L_{14} \\ 2/3L_6 & 2L_7 \end{pmatrix},$$

$$L_{18} = 2, \begin{pmatrix} -2/3L_8 - 2/3L_{12} & 2L_{15} \\ 2/3L_9 & 2L_{10} \end{pmatrix},$$

$$L_{19} = 8/3, \begin{pmatrix} 2L_{19} & 8/3L_5 \\ 0 & 0 \end{pmatrix},$$

$$L_{20} = -4, \begin{pmatrix} 4/3L_{12} - 2/3L_{16} & -2L_{13} + 2L_{20} \\ 2/3L_{17} & 2L_{18} \end{pmatrix}.$$

The four matrices $L_5, L_{12}, L_{13}, L_{19}$ span a somewhat trivial four-dimensional subalgebra consisting only of matrix sequences with zero coefficients except for the first row.

Consideration of the images $\rho(s,t)\overline{L}$ yields the following decomposition of the recursively closed vector space $\overline{L} = \bigoplus_{j=1}^{20} \mathbb{C}L_j$:

$$\rho(0,0)\mathcal{V} = \mathbb{C}L_1 \oplus \mathbb{C}L_4 \oplus \mathbb{C}L_8 \oplus \mathbb{C}L_{12} \oplus \mathbb{C}L_{16} \oplus \mathbb{C}L_{19} ,$$
$$\rho(0,1)\mathcal{V} = \mathbb{C}L_5 \oplus \mathbb{C}L_{11} \oplus \mathbb{C}L_{13} \oplus \mathbb{C}L_{14} \oplus \mathbb{C}L_{15} \oplus \mathbb{C}L_{20} ,$$
$$\rho(1,0)\mathcal{V} = \mathbb{C}L_2 \oplus \mathbb{C}L_6 \oplus \mathbb{C}L_9 \oplus \mathbb{C}L_{17} ,$$
$$\rho(1,1)\mathcal{V} = \mathbb{C}L_3 \oplus \mathbb{C}L_7 \oplus \mathbb{C}L_{10} \oplus \mathbb{C}L_{18} .$$

I ignore if the vector space $\overline{L}$ contains generators of interesting algebras or groups.

The diagonal recurrence matrix $D = D_1$ involved in $J = LDL^t$ is of

complexity 4 with recursive presentation

$$D_1 = 1, \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix}, \qquad D_2 = -1, \begin{pmatrix} D_3 & \\ & D_4 \end{pmatrix},$$

$$D_3 = 3, \begin{pmatrix} 3D_1 & \\ & 1/3D_2 \end{pmatrix}, \qquad D_4 = -1/3, \begin{pmatrix} 3D_3 & \\ & 1/3D_4 \end{pmatrix}.$$

8 $q-$binomials

Proof of Theorem 2.9 The result holds for $b = 0$ or for $a \leq b$. An induction on $a + b$ ends the proof. It splits into the four following subcases:

If $a, b \not\equiv 0 \pmod{n}$:

$$\begin{aligned}
\binom{a}{b}_q &= \omega^b \binom{a-1}{b}_\omega + \binom{a-1}{b-1}_\omega \\
&= \binom{\lfloor a/n \rfloor}{\lfloor b/n \rfloor} \left(\omega^b \binom{a-1 \pmod n}{b \pmod n}_\omega + \binom{a-1 \pmod n}{b-1 \pmod n}_\omega \right) \\
&= \binom{\lfloor a/n \rfloor}{\lfloor b/n \rfloor} \binom{a \pmod n}{b \pmod n}_\omega
\end{aligned}$$

If $a \equiv 0 \pmod{n}, b \not\equiv 0 \pmod{n}$:

$$\begin{aligned}
\binom{a}{b}_q &= \omega^b \binom{a-1}{b}_\omega + \binom{a-1}{b-1}_\omega \\
&= \binom{a/n-1}{\lfloor b/n \rfloor} \left(\omega^b \binom{n-1}{b \pmod n}_\omega + \binom{n-1}{(b-1) \pmod n}_\omega \right) \\
&= \binom{a/n-1}{\lfloor b/n \rfloor} \binom{n}{b \pmod n}_\omega
\end{aligned}$$

and $\binom{n}{b \pmod n}_\omega = 0$ since $b \not\equiv 0 \pmod{n}$ implies that it is divisible by the $n-$th cyclotomic polynomial.

If $a \not\equiv 0, b \equiv 0 \pmod{n}$:

$$\begin{aligned}
\binom{a}{b}_q &= \omega^b \binom{a-1}{b}_\omega + \binom{a-1}{b-1}_\omega \\
&= \binom{a-1}{b}_\omega + \binom{\lfloor a/n \rfloor}{b/n-1} \binom{a-1 \pmod n}{n-1}_\omega \\
&= \binom{\lfloor a/n \rfloor}{b/n} \binom{a-1 \pmod n}{0}_\omega + 0 \\
&= \binom{\lfloor a/n \rfloor}{b/n} \binom{a \pmod n}{0}_\omega
\end{aligned}$$

If $a \equiv b \equiv 0 \pmod{n}$:

$$\begin{aligned}
\binom{a}{b}_q &= \omega^b \binom{a-1}{b}_\omega + \binom{a-1}{b-1}_\omega \\
&= \binom{a/n-1}{b/n} \binom{n-1}{0}_\omega + \binom{a/n-1}{b/n-1} \binom{n-1}{n-1}_\omega \\
&= \binom{a/n}{b/n} \binom{0}{0}_\omega
\end{aligned}$$

$\square$

Remark 8.1. *I thank C. Krattenthaler for pointing out that Theorem 2.9 follows also from the classical $q-$binomial identity*

$$\prod_{j=0}^{n-1}(1 - q^j t) = \sum_{k=0}^{n}(-t)^k q^{\binom{k}{2}} \binom{n}{k}_q.$$

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Setting $n = a$ and $q = \omega$ for ω a primitive $d-$th root of 1 and equating the coefficients of t^b on both sides implies the result easily for odd d. The case of d even requires also a sign analysis.

The above identity follows by induction on n from the easy computation

$$
\prod_{j=0}^n (1 - q^j t) = (1 - q^n t) \sum_{k=0}^n (-t)^k q^{\binom{k}{2}} \binom{n}{k}_q
$$
$$
= 1 + \sum_{k=1}^{n+1} (-t)^k q^{\binom{k}{2}} \left(\binom{n}{n-k}_q + q^{n+1-k} \binom{n}{n+1-k}_q \right)
$$
$$
= \sum_{k=0}^{n+1} (-t)^k q^{\binom{k}{2}} \binom{n+1}{k}_q .
$$

Theorem 2.9 implies in particular that for $\omega = e^{2i\pi k/n}$, the complex numbers $\binom{a}{b}_\omega$, $a, b \in \mathbb{N}$ belong to the finite subset $\cup_{0 \le b \le a < n} \mathbb{R}_{\ge 0} \binom{a \pmod{n}}{b \pmod{n}}_\omega$ of real half-lines in $\mathbb{C}$.

For $n = 2$, the matrix $\binom{a}{b}_{-1}$, $0 \le a, b < 2$ is given by

$$
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} .
$$

This implies that $\binom{a}{b}_{-1} \in \mathbb{N}$ for all $a, b \in \mathbb{N}$.

For $n = 4$ we get $\binom{a}{b}_i \in \mathbb{N} \cup i\mathbb{N} \cup (1 + i)\mathbb{N}$ since we have

$$
\begin{pmatrix}
1 & & & \\
1 & 1 & & \\
1 & 1+i & 1 & \\
1 & i & i & 1
\end{pmatrix}
$$

for the matrix with coefficients $\binom{a}{b}_i$, $0 \le a, b < 4$.

8.1   Tensor products

Replacing coefficients in the ground-field $\mathbb{C}$ by multiples of a fixed matrix X of size $2^K \times 2^K$, one can consider the tensor product

$$
A \otimes X = (A[0] \otimes X, A[1] \otimes X, \dots)
$$

of $A \in \mathcal{A}$ or $A \in \mathcal{R}$ with X. Such an element has an LU decomposition $A \otimes X = (L' \otimes L_X)(U' \otimes U_X)$ involving elements of the same form if and only if we have decompositions $A = L'U' \in \mathcal{R}$ and $X = L_X U_X$.

Remark 8.2. *More generally, one can consider the quotient algebras $\mathcal{A}/\mathcal{FS}$ and $\mathcal{R}/\mathcal{FS}$ where $\mathcal{FS}$ is the ideal of all matrix sequences in $\mathcal{A}$ which involve only finitely many non-zero matrices. Elements of the quotient algebra $\mathcal{R}/\mathcal{FS}$ can be represented as linear combinations of suitable elements in $\cup_{K \in \mathbb{N}} \mathcal{R} \otimes M_{2^K \times 2^K}$ and such representations are sometimes simpler than recursive presentations of preimages in $\mathcal{R}$.*

8.2 The Beeblebrox reduction of $\binom{s+t}{s}_{-1}$

We denote by Z' the infinite symmetric matrix with coefficients $Z'_{s,t} = \chi_B(\binom{s+t}{s}_{-1})$ given by the Beeblebrox reduction of q−binomials evaluated at $q = -1$.

Theorem 2.9 and Section 8.1 imply that $Z' = L'D'(L')^t$ where

$$Z' = Z \otimes \begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix}, \ L' = L \otimes \begin{pmatrix} 1 & \\ 1 & 1 \end{pmatrix}, \ D' = D \otimes \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

(the tensor product $X \otimes M$ denotes the matrix(-sequence) obtained by re-placing a scalar entry λ of Z by the 2×2 matrix M) with Z, L, D as in the proof of Theorem 2.6.

In particular, using Theorem 2.6 and the LU decomposition of $\begin{pmatrix} 1 & 1 \\ 1 & \end{pmatrix} = \left(\chi_B(\binom{s+t}{s}_{-1}) \right)_{0 \leq s,t \leq 1}$, one can easily write down a formula for $\det(Z'(n)) \in \pm 3^{\mathbb{N}}$ with $Z'(n)$ denoting the symmetric $n \times n$ submatrix consisting of the first n rows and columns of Z'.

Remark 8.3. *The case of the matrix (with coefficients in $\{0,1\}$) obtained by reducing $\binom{s+t}{s}_{-1}$ modulo 2 yields nothing new since $\binom{s+t}{s}_{-1} \equiv \binom{s+t}{s}_1 = \binom{s+t}{s}$ (mod 2).*

8.3 Reduction modulo 2 and Beeblebrox reduction of $\binom{s+t}{s}_i$

Let M' be the symmetric matrix with coefficients

$$\psi(\binom{s+t}{s}_i) \in \{0, \pm 1, \pm x, \pm y\}, \ 0 \leq s,t$$

where

$$\psi(\xi) = \begin{cases} \gamma(\xi) & \text{if } \xi \in \mathbb{N} \\ \gamma(a)x & \text{if } \xi = ai \in i\mathbb{N} \\ \gamma(a)y & \text{if } \xi = a(1+i) \in (1+i)\mathbb{N} \end{cases}$$

where $\gamma : \mathbb{N} \longrightarrow \{0, \pm 1\}$ is either the reduction modulo 2 with values in $\{0,1\}$ or the Beeblebrox reduction χ_B.

As in section 8.2 we have $M' = L'D'(L')^t$ where

$$M' = M \otimes \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & y & x & \\ 1 & x & & \\ 1 & & & \end{pmatrix},$$

$$L' = L \otimes \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & \frac{1-x}{1-y} & 1 & \\ 1 & \frac{1}{1-y} & \frac{y-x}{x^2-2x+y} & 1 \end{pmatrix},$$

$$D' = D \otimes \begin{pmatrix} 1 & & & \\ & y-1 & & \\ & & \frac{x^2-2x+y}{1-y} & \\ & & & \frac{-x^2}{x^2-2x+y} \end{pmatrix}.$$

M, L, D are given by the matrices P, L, D, respectively Z, L, D occuring in the proof of Theorem 2.1, respectively 2.6, if γ is the reduction modulo 2, respectively the Beeblebrox reduction.

It follows that the determinant $\det(M'(n))$ of the finite matrix $M'(n)$ consisting of the first n rows and columns of M' is of the form

$$\pm 3^{\mathbb{N}} x^{2\mathbb{N}} (y-1)^{\{0,1\}} (x^2 - 2x + y)^{\{0,1\}}$$

with powers of 3 only involved if γ is the Beeblebrox reduction. The factor $(y-1)$ appears if and only if $n \equiv 2 \pmod 4$ and the factor $(x^2 - 2x + y)$ appears if and only if $n \equiv 3 \pmod 4$. Using Theorem 2.1, respectively Theorem 2.6, it is easy to write down a formula for $\det(M'(n))$.

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